

Mean excitation numbers due to anti-rotating term (MENDART) in cavity QED under Lindbladian dephasing

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We study the photon generation from arbitrary initial state in cavity QED due to the combined action of the anti-rotating term present in the Rabi Hamiltonian and Lindblad-type dephasing. We obtain a simple set of differential equations describing this process and deduce useful formulae for the moments of the photon number operator, demonstrating analytically that the average photon number increases linearly with time in the asymptotic limit.

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In the 2008-th paper by *Werlang et al.* [1] a puzzling quantum effect was noticed from numerical simulations: when a two-level atom interacts with a single mode of the radiation field in a cavity by means of the Rabi Hamiltonian, while subject to standard Markovian dephasing mechanism, the average intracavity photon number exhibits a linear growth with time. Such asymptotic photon generation due to decoherence occurs because for pure dephasing processes the environment may be viewed as a unmonitored detector making random nondemolition measurements of the number of quanta in the atom-field system [2, 3], whilst in [4–6] it was shown that nondemolition measurements can pump energy into the system via the destruction of quantum coherence provided the anti-rotating term is kept in the light-matter interaction Hamiltonian (i.e., without performing the Rotating Wave Approximation [7]). Besides, the pure dephasing reservoirs always possess a finite temperature (see, e.g. [2, 3] for microscopic deduction) and hence store an infinite amount of energy, so the additional system energy is continuously supplied by the environment and the First Law of Thermodynamics is not violated (for the discussion concerning the Second Law of Thermodynamics in systems subject to frequent quantum measurements see [5]).

Although the phenomenon of photon generation due to decoherence was explained qualitatively in [1, 8], no satisfactory analysis was carried out to derive analytically whether for the pure Markovian dephasing the average photon number de facto increases linearly with time and whether this growth saturates for large times. So the aim of this paper is to investigate analytically the behavior of Mean Excitation Numbers due to Anti-Rotating Term (MENDART), such as mean photon number and its variance or atomic excitation probability, and investigate their asymptotic characteristics in the simplest case of Markovian dephasing. We shall show that for any initial state in the asymptotic limit the mean photon number $\langle n \rangle$ indeed increases linearly with time, the average value of the photon number second moment $\langle n^2 \rangle$ grows quadratically with time, and the atomic excitation probability P_e attains a constant value. So this paper provides the missing mathematical explanation for the phenomenon of steady photon generation due to Lindblad-

type decoherence in the presence of the anti-rotating term.

Our starting point is the Markovian master equation for the density matrix ρ that takes into account both the atomic and cavity field phase-damping (dephasing) [2, 3, 8, 9]

$$\dot{\rho} = -i[H, \rho] + \frac{\gamma_a}{2}(\sigma_z \rho \sigma_z - \rho) + \gamma_c(2n\rho n - n^2\rho - \rho n^2), \quad (1)$$

where γ_a (γ_c) is the atomic (cavity) dephasing rate and H is the Rabi Hamiltonian [10, 11] (we set $\hbar = 1$)

$$H = \omega n + \frac{\Omega}{2}\sigma_z + g(a + a^\dagger)(\sigma_+ + \sigma_-) \quad (2)$$

that includes the anti-rotating term ($a\sigma_- + a^\dagger\sigma_+$). Here a and a^\dagger are the cavity annihilation and creation operators, $n \equiv a^\dagger a$ is the photon number operator, and ω , Ω and g are the cavity frequency, the atomic transition frequency and the atom-field coupling constant, respectively. The Pauli operators are defined as $\sigma_- = |g\rangle\langle e|$, $\sigma_+ = |e\rangle\langle g|$ and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, so that kets $|g\rangle$ and $|e\rangle$ can be interpreted as atomic ground and excited states, respectively.

Expanding the density matrix in the Fock basis as

$$\rho = \sum_{n,m=0}^{\infty} (a_{n,m}|g,n\rangle\langle g,m| + b_{n,m}|e,n\rangle\langle e,m| + c_{n,m}|g,n\rangle\langle e,m| + c_{m,n}^*|e,n\rangle\langle g,m|), \quad (3)$$

where $a_{n,m}$, $b_{n,m}$ and $c_{n,m}$ are time-dependent coefficients, we obtain the exact set coupled differential equations (the prime stands for the time derivative)

$$a'_{n,m} = i[\omega(m-n) + i\gamma_c(n-m)^2]a_{n,m} + ig(\sqrt{m}c_{n,m-1} - \sqrt{n}c_{m,n-1}^* + \sqrt{m+1}c_{n,m+1} - \sqrt{n+1}c_{m,n+1}^*)(4)$$

$$b'_{n,m} = i[\omega(m-n) + i\gamma_c(n-m)^2]b_{n,m} + ig(\sqrt{m}c_{m-1,n}^* - \sqrt{n}c_{n-1,m} + \sqrt{m+1}c_{m+1,n}^* - \sqrt{n+1}c_{n+1,m})(5)$$

$$c'_{n,m} = if_{n,m}c_{n,m} + ig(\sqrt{m}a_{n,m-1} - \sqrt{n+1}b_{n+1,m} + \sqrt{m+1}a_{n,m+1} - \sqrt{n}b_{n-1,m}), \quad (6)$$

where $f_{n,m} \equiv \omega(m-n) + \Omega + i[\gamma_a + \gamma_c(n-m)^2]$. In the strong dephasing limit, $(\gamma_a + \gamma_c) \gtrsim |g|$, one expects on physical ground that due to the decoherence the terms $c_{n,m}$ rapidly attain some constant values, so assuming that $c'_{n,m} = 0$ we get

$$c_{n,m} = \frac{g}{f_{n,m}} (\sqrt{n+1}b_{n+1,m} - \sqrt{m}a_{n,m-1} + \sqrt{n}b_{n-1,m} - \sqrt{m+1}a_{n,m+1}). \quad (7)$$

Now we substitute the expression for $c_{n,m}$ back into equations (4)–(5)[19] and define new coefficients $\tilde{a}_{n,m} = e^{-i\omega t(m-n)}a_{n,m}$ and $\tilde{b}_{n,m} = e^{-i\omega t(m-n)}b_{n,m}$ that are slowly varying functions of time. Assuming that $|g| \ll \omega, \Omega$ (a condition that holds in cavity QED experiments unless the so-called ‘ultra-strong coupling regime’ [12, 13] is achieved) we neglect the rapidly oscillating terms and obtain the following effective differential equations for the diagonal probability amplitudes

$$\tilde{a}'_n = -[(v_1 + v_2)n + v_2]\tilde{a}_n + [v_1n\tilde{b}_{n-1} + v_2(n+1)\tilde{b}_{n+1}] \quad (8)$$

$$\tilde{b}'_n = -[(v_1 + v_2)n + v_1]\tilde{b}_n + [v_2na_{n-1} + v_1(n+1)\tilde{a}_{n+1}], \quad (9)$$

where $\tilde{a}_n \equiv \tilde{a}_{n,n}$, $\tilde{b}_n \equiv \tilde{b}_{n,n}$ and we defined coefficients

$$v_1 = \frac{2\gamma g^2}{(\omega - \Omega)^2 + \gamma^2}, \quad v_2 = \frac{2\gamma g^2}{(\omega + \Omega)^2 + \gamma^2} \quad (10)$$

with $\gamma \equiv \gamma_c + \gamma_a$ standing for the total dephasing rate.

One can easily verify that the normalization condition is maintained, $\sum_{n=0}^{\infty}(\tilde{a}'_n + \tilde{b}'_n) = 0$, so the equations (8)–(9) are consistent and lead to the following coupled differential equations for the low-order MENDART

$$\langle n(t) \rangle' = v_2 + (v_1 - v_2) [P_e(t) + \langle n\sigma_z(t) \rangle] \quad (11)$$

$$P_e(t)' = v_2 - (v_1 + v_2) [P_e(t) + \langle n\sigma_z(t) \rangle] \quad (12)$$

$$\langle n\sigma_z(t) \rangle' = v_2 - 2(v_1 - v_2) \langle n(t) \rangle - (v_1 + v_2) [P_e(t) + \langle n\sigma_z(t) \rangle + 2\langle n^2\sigma_z(t) \rangle] \quad (13)$$

$$\langle n^2(t) \rangle' = 2(\omega^2 + \Omega^2 + \gamma^2)^{-1} \times [\gamma g^2 (1 + 4\langle n(t) \rangle) - \omega\Omega \langle n\sigma_z(t) \rangle']. \quad (14)$$

These equations cannot be solved analytically due to the coupling to the dynamical variable $\langle n^2\sigma_z(t) \rangle$ which obeys another differential equation.

However, one can deduce the *general* formula for the average photon number $\langle n(t) \rangle$ by noticing the similarity in the last terms of equations (11) and (12). One gets

$$\langle n(t) \rangle - \langle n(0) \rangle = \frac{2}{\omega^2 + \Omega^2 + \gamma^2} \{g^2\gamma t - \omega\Omega[P_e(t) - P_e(0)]\}, \quad (15)$$

where $P_e(t) \leq 1$ is still an unknown function of time[20]. Furthermore, in the asymptotic regime $t \rightarrow \infty$ we expect from the equation (12) that P_e attains a constant value. Imposing $\lim_{t \rightarrow \infty} P_e(t)' = 0$ we obtain from (12)–(13)

$$\lim_{t \rightarrow \infty} [P_e(t) + \langle n\sigma_z(t) \rangle] = \frac{1}{2} - \frac{\omega\Omega}{\omega^2 + \Omega^2 + \gamma^2} \quad (16)$$

$$\lim_{t \rightarrow \infty} \langle n\sigma_z(t) \rangle' = 0 \quad (17)$$

$$\lim_{t \rightarrow \infty} \langle n^2\sigma_z(t) \rangle = -\frac{2\omega\Omega}{\omega^2 + \Omega^2 + \gamma^2} \lim_{t \rightarrow \infty} \langle n(t) \rangle \quad (18)$$

and from equation (14) we get

$$\lim_{t \rightarrow \infty} \langle n^2(t) \rangle' = \frac{2\gamma g^2}{\omega^2 + \Omega^2 + \gamma^2} \left[1 + 4 \lim_{t \rightarrow \infty} \langle n(t) \rangle \right]. \quad (19)$$

Hence, in the asymptotic regime $t \rightarrow \infty$ we have the following rules for the Asymptotic MENDART (AMENDART, as coined in [9]) for any initial state: **a)** $\langle n(t) \rangle$ and $-\langle n^2\sigma_z(t) \rangle$ increase *linearly* with time; **b)** $\langle n^2(t) \rangle$ increases *quadratically* with time; **c)** $P_e(t)$ and $\langle n\sigma_z(t) \rangle$ attain *constant* values. We solved numerically the effective differential equations (8)–(9) and verified that the formula (15) is correct for all times, thereby accounting for the linear growth of $\langle n(t) \rangle$ noticed in [1] from numerical data, while the equations (16)–(19) agree with the numerical results in the asymptotic regime.

In the figures 1–3 we compare the exact dynamics resultant from the original differential equations (4)–(6) to the effective dynamics governed by the simplified equations (8)–(9) for the parameters $\omega = 1$, $g = 4 \times 10^{-2}$, $\gamma_a = 2g$ and $\gamma_c = 0$. In the figure 1 (2) we consider the resonant regime $\Omega = \omega$ (dispersive regime $\Omega = \omega - 20g$) for the initial zero-excitation state $|g, 0\rangle$. We plot the dynamical behavior of observables $\langle n \rangle$, $\langle n^2 \rangle$, $\langle n^2\sigma_z \rangle$, $\langle n\sigma_z \rangle$ and P_e calculated from the original differential equations (4)–(6). Within the thickness of the lines these curves are indistinguishable from the graphs resulting out of the effective equations (8)–(9). To exemplify the difference between the original and effective differential equations, we show the zoom for the behavior of $\langle n^2 \rangle$ at initial times: the solid line depicts the exact dynamics, and the dashed line – the effective one. The observed discrepancies are quite small and appear because $c'_{n,m}$ does not becomes zero instantly as was assumed in our analysis; nevertheless, these minor differences are irrelevant regarding the asymptotic behavior. In the figures we also show the photon number distributions calculated numerically at the time $gt = 300$ according to the original differential equations (bars) and the effective ones (dots). Once again, the agreement is excellent.

In the figure 3 we consider the initial state $\rho(0) = \rho_{therm} \otimes |e\rangle\langle e|$, where ρ_{therm} is the thermal state of the Electromagnetic field whose photon number distribution is $p_n = \bar{n}^n / (\bar{n} + 1)^{n+1}$, where \bar{n} is the average photon

number. We set $\Omega = \omega$, $\bar{n} = 0.3$ and show the asymptotic behavior of $\langle n \rangle$, $\langle n^2 \rangle$ and $\langle n^2 \sigma_z \rangle$ obtained from the original differential equations (figure 3a) and the zoom of $\langle n \rangle$ and P_e for initial times (figure 3b, where the dashed lines represent the solutions of the effective differential equations). We see that asymptotically the behavior agrees with equations (15)–(19), although the transient dynamics cannot be reliably described by the equations (8)–(9).

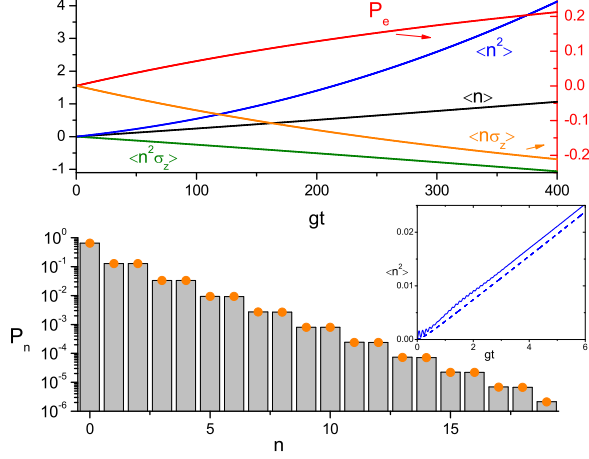


FIG. 1: Exact and effective dynamical behavior of principal observables and the photon statistics for the time $gt = 300$ in the resonant regime, $\Omega = \omega$.

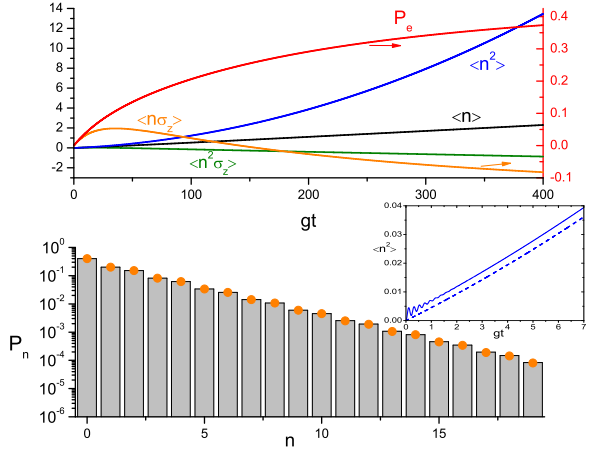


FIG. 2: Same as figure 1 in the dispersive regime, $\Omega = \omega - 20g$.

Regarding the practical observation of the asymptotic linear photon growth inside the cavity due to decoherence, it seems quite unlikely in current cavity or circuit

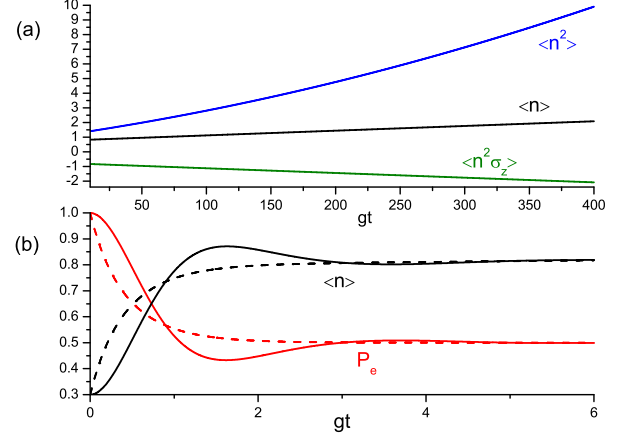


FIG. 3: Behavior of $\langle n \rangle$, $\langle n^2 \rangle$, $\langle n^2 \sigma_z \rangle$ and P_e in (a) the asymptotic regime and (b) during the transient regime for small times. The initial state is $\rho_{therm} \otimes |e\rangle\langle e|$ with average photon number $\langle n(0) \rangle = 0.3$.

QED implementations because the photons (and atomic excitations) would be lost due to radiative and nonradiative relaxation processes. As example, let us consider the current state of the art circuit QED implementations. The typical parameters are [15]: $\omega \sim \Omega \sim 8$ GHz and $g \sim 0.3$ GHz, while the dephasing rate is of the order of $\gamma_a \sim 1$ MHz, although it can be made large at will (usually one desires to decrease γ and not to increase it). Considering a high value for the total dephasing rate $\gamma \sim 1$ GHz the resulting asymptotic photon growth rate due to dephasing would be ~ 1 MHz. This value is of the same order of magnitude as the cavity relaxation rate for a rather high cavity quality factor $Q \sim 10^4$, so $\langle n(t) \rangle$ would saturate at some (small) value instead of showing an asymptotic growth, as calculated explicitly in [8, 9, 16] for standard quantum optical master equation. Some photons escape to the outside world via radiative dissipation channel so they could be ultimately detected outside the cavity, but in this case different models predict different photon emission rates depending on assumptions made about the reservoir [17, 18] (in particular whether it is Markovian or not).

Recently a more sophisticated microscopic model was developed for deducing the master equation in the presence of the anti-rotating term, valid in a specific regime of parameters [13]. According to that model, the phenomenon of dephasing-induced generation of photons is greatly exaggerated by the Lindblad-type master equation (1), and instead of the linear asymptotic growth the average photon number saturates at some value that strongly depends on the reservoir spectral density [13]. Nevertheless, the very phenomenon of photon generation due to decoherence persists and our formulae provide the upper bound for the photon generation rate. From the

qualitative viewpoint, in realistic (lossy) cavity QED architectures this phenomenon would lead to a parameter-dependent heating of the system slightly above the thermal equilibrium values [9, 16], depending on the atom-field detuning, coupling strength and the dephasing rate.

In summary, we obtained simplified differential equations describing the process of photon generation (from vacuum or any other state) due to the combined action of the anti-rotating term and the standard Lindbladian dephasing in Markovian cavity QED, whose validity was confirmed by extensive numerical simulations. From these equations we deduced analytical formulae describing the overall behavior of MENDART for arbitrary ini-

tial state, demonstrating that asymptotically the mean photon number $\langle n \rangle$ increases linearly with time at the rate $2\gamma g^2(\omega^2 + \Omega^2 + \gamma^2)^{-1}$, $\langle n^2 \rangle$ grows quadratically with time and the atomic excitation probability attains a constant value.

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 - [19] Actually, one must have in mind that the equation (7) only holds after sufficient amount of time, but such nuances are not relevant when one is interested in the asymptotic behavior.
 - [20] Notice that the obtained asymptotic photon generation rate $2\gamma g^2(\omega^2 + \Omega^2 + \gamma^2)^{-1}$ resembles the approximate formula obtained in [14] [namely $2\gamma g^2((\omega + \Omega)^2 + \gamma^2)^{-1}$], although in that paper the mathematical approach was oversimplified.